The Dyadic Green's Functions for Cylindrical Waveguides and Cavities

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***Abstract*—Four dyadics are derived to find the electric and magnetic fields generated by a given distribution of electric and magnetic (including aperture) currents in cylindrical waveguides and cavities of arbitrary cross sections. Two sets of vectors are used to form the dyadics: one set is an expansion of "electric field" vectors, and the other is an expansion of "magnetic field" vectors. Explicit expressions in terms of TE and TM modes are obtained for the resulting electric and magnetic fields. Inside the source regions there are additional components proportional either to the axial components of the current densities (waveguides), or to the current densities vectors (cavities).**

I. Introduction

THE FUNDAMENTALS of the electromagnetic Green's function theory have been developed by Stratton [1], Morse and Feshbach [2], Tai [3], and Felsen and Marcuvitz [4]. Expressions for the Green's function that determine the electric field generated by electric currents in rectangular waveguides and cavities have been obtained by Collin [5], Rahmat-Samii [6], and Tai and Rozenfeld [7]. Recent research in waveguides (e.g., [8], [9]) show that there is a need for a straightforward and comprehensive way to obtain the dyadics that determine both electric and magnetic fields generated by electric and magnetic currents (including aperture) in cylindrical waveguides and cavities of arbitrary cross section, and thus overcome "the deficiencies of the Green's functions in the waveguide region" [8, p. 457].

This paper presents a detailed derivation of these dyadics. Explicit expressions are obtained for the electric and magnetic fields generated by a given (or assumed) distribution of electric and magnetic current densities. The fields outside the source region are described by expansions of TE and TM modes. The waveguide fields inside the source region are also expansions of TE and TM modes with additional axial components proportional to the axial components of the current densities. The fields inside the source regions in cavities have additional vector components proportional to the full current densities.

II. Dyadic Green's Functions for the Wave Equation

Consider a cylindrical infinite waveguide (or a cavity) with perfectly conducting walls. The cross section is S with a boundary $\mathfrak{L}$. The guide (cavity) is filled with an homogeneous isotropic medium, and is driven by an arbitrary distribution of electric and magnetic currents located inside a finite volume V. The electromagnetic field in the waveguide (cavity) is determined by the Maxwell equations

$$\nabla\times\boldsymbol{E}=-j\omega\mu\boldsymbol{H}-\boldsymbol{J}_m$$
$$\nabla\times\boldsymbol{H}=j\omega\epsilon\boldsymbol{E}+\boldsymbol{J}_e \tag{1}$$

and by the homogeneous boundary conditions at the walls, as well as the "radiation condition" at $|z|\to\infty$ in the waveguide.

Equations (1) are solved by the standard methods of vector analysis

$$\boldsymbol{E}(\boldsymbol{r})=\int_V\bar{G}_e(\boldsymbol{r},\boldsymbol{r}')\Big[(j\omega\epsilon)^{-1}(\bar{I}k^2+\nabla'\nabla')\boldsymbol{J}_e(\boldsymbol{r}')-\nabla'\times\boldsymbol{J}_m(\boldsymbol{r}')\Big]dV'$$
$$\boldsymbol{H}(\boldsymbol{r})=\int_V\bar{G}_m(\boldsymbol{r},\boldsymbol{r}')\Big[(j\omega\mu)^{-1}(\bar{I}k^2+\nabla'\nabla')\boldsymbol{J}_m(\boldsymbol{r}')+\nabla'\times\boldsymbol{J}_e(\boldsymbol{r}')\Big]dV' \tag{2}$$

where the primes on the dels denote differentiation with respect to the primed coordinates $\boldsymbol{r}'$ of the source point

$$k^2=\omega^2\epsilon\mu. \tag{3}$$

$\bar{G}_e$ and $\bar{G}_m$ are the "electric" and "magnetic" dyadic Green's functions which satisfy the equation

$$(\nabla^2+k^2)\bar{G}=-\bar{I}\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{4}$$

$\bar{I}$ is the unity dyadic or idemfactor. The vectors that form the dyadic $\bar{G}_e$ have zero tangential components on the metallic walls, while the vectors that form $\bar{G}_m$ have zero normal components on the walls.

Another well-known solution of (1) is obtained using the Hertz potentials (e.g., [1, ch. 8]) $\boldsymbol{Z}_e$ and $\boldsymbol{Z}_m$

$$\boldsymbol{E}=(\bar{I}k^2+\nabla\nabla)\boldsymbol{Z}_e-j\omega\mu\nabla\times\boldsymbol{Z}_m$$
$$H=(\bar{I}k^2+\nabla\nabla)\boldsymbol{Z}_m+j\omega\epsilon\nabla\times\boldsymbol{Z}_e \tag{5}$$

where

$$\boldsymbol{Z}_e(\boldsymbol{r})=(j\omega\epsilon)^{-1}\int_V\bar{G}_e(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_e(\boldsymbol{r}')\,dV'$$
$$\boldsymbol{Z}_m(\boldsymbol{r})=(j\omega\mu)^{-1}\int_V\bar{G}_m(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_m(\boldsymbol{r}')\,dV'. \tag{6}$$

Now, we substitute (6) into (5), taking into account that the differential operators could be brought inside the

Manuscript received December 5, 1979; revised March 19, 1980.

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0018-9480/80/0800-0894$00.75

integral signs since they affect only the observation point coordinates $\boldsymbol{r}$, while the integration is over the source point coordinates $\boldsymbol{r}'$. The final expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ have the form

$$\boldsymbol{E}(\boldsymbol{r})=(j\omega\epsilon)^{-1}\int_V \bar{Y}_e(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_e(\boldsymbol{r}')\,dV'-\int_V \bar{X}_m(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_m(\boldsymbol{r}')\,dV'$$

$$\boldsymbol{H}(\boldsymbol{r})=(j\omega\mu)^{-1}\int_V \bar{Y}_m(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_m(\boldsymbol{r}')\,dV'+\int_V \bar{X}_e(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}_e(\boldsymbol{r}')\,dV' \tag{7}$$

where

$$\bar{Y}_e=(\bar{I}k^2+\nabla\nabla)\bar{G}_e \qquad \bar{X}_m=\nabla\times\bar{G}_m,\text{etc.} \tag{8}$$

are the dyadics that determine the electromagnetic field generated by the current densities $\boldsymbol{J}_e$ and $\boldsymbol{J}_m$. It is easy to show by direct substitution that the fields (7) satisfy Maxwell's equations (1) and the divergence equations

$$\nabla\cdot\boldsymbol{E}=-(j\omega\epsilon)^{-1}\nabla\cdot\boldsymbol{J}_e$$

$$\nabla\cdot\boldsymbol{H}=-(j\omega\mu)^{-1}\nabla\cdot\boldsymbol{J}_m \tag{9}$$

which follow from (1), and require that both dyadic Green's functions satisfy the condition

$$\nabla\cdot\bar{G}\neq 0. \tag{10a}$$

The fields (7) must also satisfy the boundary conditions imposed by the conducting walls, and must be identical with (2). Integrating (2) by parts over the finite volume of the source region and comparing the result with (7) and (8), we obtain the following relationships between the dyadics $\bar{G}_e$ and $\bar{G}_m$

$$\nabla'\times\bar{G}_e=\nabla\times\bar{G}_m \qquad \nabla'\times\bar{G}_m=\nabla\times\bar{G}_e. \tag{10b}$$

Equations (10) and the boundary conditions imposed on the vectors that form the dyadics G_e and G_m determine two complete sets of vectors eigenfunctions or eigenvectors: one set with zero tangential components on the walls ("electric" type), and the second set with zero normal components on the walls ("magnetic" type). The eigenvectors of both sets must satisfy the vector Helmholtz equation

$$\nabla^2\boldsymbol{A}_\nu+k_\nu^2\boldsymbol{A}_\nu=0 \tag{11}$$

where k_ν^2 is the eigenvalue, and $\boldsymbol{A}_\nu$ is the corresponding eigenvector.

The "electric" eigenvectors are well known [2]:

$$\boldsymbol{L}_\nu=k_\nu^{-1}\nabla f_{L\nu} \qquad \boldsymbol{M}_\nu=k_\nu^{-1}\nabla\times(\hat{z}f_{M\nu})$$

$$\boldsymbol{N}_\nu=k_\nu^{-2}\nabla\times\nabla\times(\hat{z}f_{N\nu}) \tag{12}$$

where the caps denote a unit vector, and z is the axis of the waveguide (cavity). The eigenvectors (12) are orthogonal, i.e.,

$$\int_{V_0}\boldsymbol{L}_\mu^*\cdot\boldsymbol{M}_\nu\,dV=\int_{V_0}\boldsymbol{L}_\mu^*\cdot\boldsymbol{N}_\nu\,dV=0$$

$$\int_{V_0}\boldsymbol{L}_\mu^*\cdot\boldsymbol{L}_\nu\,dV=W_{L\nu}^{-1}\delta_{\mu\nu},\ \text{etc.} \tag{13}$$

where V_0 is the volume of the cavity, or of any part of the waveguide between two arbitrary chosen cross sections located at z_1 and z_2. $W_{L\nu}$, $W_{M\nu}$, and $W_{N\nu}$ are the (inverse) normalization constants. The star is used to denote complex conjugate values.

The "completeness relation" for the "electric" eigenvectors is

$$\sum_\nu\left[W_{L\nu}\boldsymbol{L}_\nu(\boldsymbol{r})\boldsymbol{L}_\nu^*(\boldsymbol{r}')+W_{M\nu}\boldsymbol{M}_\nu(\boldsymbol{r})\boldsymbol{M}_\nu^*(\boldsymbol{r}')+W_{N\nu}\boldsymbol{N}_\nu(\boldsymbol{r})\boldsymbol{N}_\nu^*(\boldsymbol{r}')\right]=\bar{I}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{14}$$

The "magnetic" eigenvectors are found with the help of (10)

$$\boldsymbol{C}_\nu=k_\nu^{-1}\nabla f_{C\nu} \qquad \boldsymbol{F}_\nu=k_\nu^{-1}\nabla\times\boldsymbol{N}_\nu,$$

$$\boldsymbol{K}_\nu=k_\nu^{-1}\nabla\times\boldsymbol{M}_\nu. \tag{15}$$

The eigenvectors (15) are orthogonal and also form a complete set.

The vectors $\boldsymbol{N}_\nu$ and $\boldsymbol{F}_\nu$ are the electric and magnetic fields of TM modes, $\boldsymbol{M}_\nu$ and $\boldsymbol{K}_\nu$ are the fields of TE modes, and $\boldsymbol{L}_\nu$ and $\boldsymbol{C}_\nu$ are curlless fields generated inside the source region.

All the scalar functions in (12) and (15) satisfy the Helmholtz equation

$$(\nabla^2+k_\nu^2)f_\nu=0 \tag{16}$$

with boundary conditions that are different for waveguides and cavities.

III. Green's Functions for Waveguides

In order to satisfy the boundary conditions for the "electric" and "magnetic" eigenvectors components defined by (12) and (15), the scalar eigenfunctions f_ν must satisfy the following boundary conditions on the waveguide cross section boundary: the functions $f_{L\nu}$ and $f_{N\nu}$, and the normal (to $\mathfrak{L}$) derivatives of $f_{C\nu}$ and $f_{M\nu}$ must be zero. Thus there are only two independent sets of scalar eigenfunctions

$$f_{L\nu}^{\pm}=f_{N\nu}^{\pm}=f_\nu(\rho)\exp(\mp j\beta_\nu z)$$

$$f_{M\nu}^{\pm}=f_{C\nu}^{\pm}=\psi_\nu(\rho)\exp(\mp j\gamma_\nu z) \tag{17}$$

which satisfy (16) with a continuous distribution of eigenvalues determined by the operating frequency (3); $\boldsymbol{\rho}$ is the two-dimensional coordinate in the waveguide cross section. The propagation constants in (17) are

$$\beta_\nu=+\left(k^2-\xi_\nu^2\right)^{1/2}$$

$$\gamma_\nu=+\left(k^2-\eta_\nu^2\right)^{1/2} \tag{18}$$

where ξ_ν^2 and η_ν^2 are the eigenvalues of the two-dimensional Helmholtz equations

$$(\nabla^2+\xi_\nu^2)f_\nu=0 \qquad (\nabla^2+\eta_\nu^2)\psi_\nu=0$$

$$\left(\frac{\partial}{\partial z}f_\nu=0,\ \frac{\partial}{\partial z}\psi_\nu=0\right) \tag{19}$$

with the boundary conditions on $\mathfrak{L}$: $f_\nu=0$, $\partial\psi_\nu/\partial n=0$. For evanescent (decaying) waves, i.e., for $|\xi_\nu|>k$ or $|\eta_\nu|>k$, the propagation constants $j\beta_\nu$ and $j\gamma_\nu$ in (17) and in all the following formulas must be replaced by $|\beta_\nu|$ and $|\gamma_\nu|$.

Now we need two new sets of eigenvectors. The "electric" eigenvectors are

$$L_\nu^\pm(r) = P_\nu^\pm(\rho)\exp(\mp j\beta_\nu z) \quad \text{(curlless)}$$
$$M_\nu^\pm(r) = R_\nu^\pm(\rho)\exp(\mp j\gamma_\nu z) \quad \text{(TE)}$$
$$N_\nu^\pm(r) = Q_\nu^\pm(\rho)\exp(\mp j\beta_\nu z) \quad \text{(TM)}. \tag{20}$$

The "magnetic" eigenvectors are

$$C_\nu^\pm(r) = S_\nu^\pm(\rho)\exp(\mp j\gamma_\nu z) \quad \text{(curlless)}$$
$$K_\nu^\pm(r) = T_\nu^\pm(\rho)\exp(\mp j\gamma_\nu z) \quad \text{(TE)}$$
$$F_\nu^\pm(r) = U_\nu^\pm(\rho)\exp(\mp j\beta_\nu z) \quad \text{(TM)}. \tag{21}$$

All the eigenvectors satisfy the condition

$$A^- = (A^+)^*.$$

These new eigenvectors are related to the scalar eigenfunctions by the following expressions which are derived after the substitution of (17) and (20) into (12), and of (17) and (21) into (15)

$$P_\nu^\pm = k^{-1}(\nabla \mp \hat{z}j\beta_\nu)f_\nu \quad \text{(curlless)}$$
$$Q_\nu^\pm = k^{-2}(\mp j\beta_\nu\nabla + \hat{z}\xi_\nu^2)f_\nu \quad \text{(TM)}$$
$$R_\nu^\pm = k^{-1}(\nabla\psi_\nu \times \hat{z}) \quad \text{(TE)} \tag{22}$$

and

$$S_\nu^\pm = k^{-1}(\nabla \mp \hat{z}j\gamma_\nu)\psi_\nu \quad \text{(curlless)}$$
$$U_\nu^\pm = k^{-1}(\nabla f_\nu \times \hat{z}) \quad \text{(TM)}$$
$$T_\nu^\pm = k^{-2}(\mp j\gamma_\nu\nabla + \hat{z}\eta_\nu^2)\psi_\nu \quad \text{(TE)}. \tag{23}$$

The "completeness relation" for the "electric" eigenvectors is

$$\sum_\nu \big[W_{P\nu}P_\nu(\rho)P_\nu^*(\rho') + W_{Q\nu}Q_\nu(\rho)Q_\nu^*(\rho') + W_{R\nu}R_\nu(\rho)R_\nu^*(\rho')\big] = \bar{I}\delta(\rho-\rho'). \tag{24}$$

The "completeness relation" for the "magnetic" eigenvectors is analogous to (24).

The waveguide dyadic Green's functions which satisfy (4) and the needed boundary conditions are

$$\bar{G}_e(r,r') = \sum_\nu \{[W_{P\nu}P_\nu(\rho)P_\nu^*(\rho') + W_{Q\nu}Q_\nu(\rho)Q_\nu^*(\rho')]\zeta_\nu(\beta,z,z') + W_{R\nu}R_\nu(\rho)R_\nu^*(\rho')\zeta_\nu(\gamma,z,z')\}$$
$$\bar{G}_m(r,r') = \sum_\nu \{[W_{S\nu}S_\nu(\rho)S_\nu^*(\rho') + W_{T\nu}T_\nu(\rho)T_\nu^*(\rho')]\zeta_\nu(\gamma,z,z') + W_{U\nu}U_\nu(\rho)U_\nu^*(\rho')\zeta_\nu(\beta,z,z')\} \tag{25}$$

where

$$\zeta_\nu(\beta,z,z') = (2j\beta_\nu)^{-1}(\lambda_{\beta\nu}^+ + \lambda_{\beta\nu}^-)$$
$$\lambda_{\beta\nu}^+ = u(z-z')\exp[-j\beta_\nu(z-z')]$$
$$\lambda_{\beta\nu}^- = u(z'-z)\exp[-j\beta_\nu(z'-z)] \tag{26}$$

$$u(x) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$$

$u(x)$ is Heaviside's unit step function which is widely used, together with its derivatives, in undergraduate textbooks [10].

The derivatives of (26) are easily found

$$\zeta_\nu' = \partial\zeta_\nu/\partial z = \tfrac{1}{2}(-\lambda_{\beta\nu}^+ + \lambda_{\beta\nu}^-) \qquad \zeta_\nu'' = -\beta_\nu^2\zeta_\nu - \delta(z-z'). \tag{27}$$

The function $\zeta_\nu(\gamma,z,z')$ and its derivatives are similar to (26) and (27). The normalization constants are

$$W_{P\nu} = W_{f\nu},\ W_{Q\nu} = W_{U\nu} = W_{N\nu} = W_{F\nu} = k^2\xi_\nu^{-2}W_{f\nu}$$
$$W_{S\nu} = W_{\psi\nu},\ W_{T\nu} = W_{R\nu} = W_{M\nu} = W_{K\nu} = k^2\eta_\nu^{-2}W_{\psi\nu} \tag{28}$$

where

$$W_{f\nu}^{-1} = \int_S |f_\nu|^2 da$$
$$W_{\psi\nu}^{-1} = \int_S |\psi_\nu|^2 da \tag{29}$$

are the normalization constants of the scalar eigenfunctions.

The dyadics $\bar{Y}(r,r')$ and $\bar{X}(r,r')$ of (7) for a generalized cylindrical waveguide can now be written in the form

$$\bar{Y}_e = \sum_\nu [\Gamma_{P\nu}P_\nu^*(\rho') + \Gamma_{Q\nu}Q_\nu^*(\rho') + \Gamma_{R\nu}R_\nu^*(\rho')]$$
$$\bar{Y}_m = \sum_\nu [\Gamma_{S\nu}S_\nu^*(\rho') + \Gamma_{T\nu}T_\nu^*(\rho') + \Gamma_{U\nu}U_\nu^*(\rho')] \tag{30}$$
$$\bar{X}_e = \sum_\nu [\Pi_{P\nu}P_\nu^*(\rho') + \Pi_{Q\nu}Q_\nu^*(\rho') + \Pi_{R\nu}R_\nu^*(\rho')]$$
$$\bar{X}_m = \sum_\nu [\Pi_{S\nu}S_\nu^*(\rho') + \Pi_{T\nu}T_\nu^*(\rho') + \Pi_{U\nu}U_\nu^*(\rho')]. \tag{31}$$

The vectors Γ and Π are determined according to (8) by the relations

$$\Gamma_A = W_A(\bar{I}k^2 + \nabla\nabla)[A(\rho)\zeta(z,z')] \tag{32}$$
$$\Pi_A = W_A\nabla\times[A(\rho)\zeta(z,z')]. \tag{33}$$

For any vector $A(\rho)$ that does not depend on the longitudinal coordinate z, expressions (32), (33) yield

$$\Gamma_A = W_A\{\zeta[k^2A + \nabla(\nabla\cdot A) - \hat{z}\beta^2A_z] + \zeta'[\nabla A_z + \hat{z}(\nabla\cdot A)] - \hat{z}A_z\delta(z-z')\} \tag{34}$$
$$\Pi_A = W_A(\zeta\nabla\times A + \zeta'\hat{z}\times A) \tag{35}$$

where ζ and ζ' are defined by (26) and (27).

The substitution of (34) and (35) into (30) and (31) yields the following expressions for the dyadic Green's functions $\bar{Y}_e(r,r')$ and $\bar{X}_m(r,r')$ that determine the electric field in (7)

$$\bar{Y}_e(r,r') = \sum_\nu \Big\{ W_{f\nu}\Big[(\beta_\nu^2\xi_\nu^{-2}\nabla f_\nu\nabla' f_\nu' + \xi_\nu^2\hat{z}\hat{z}f_\nu f_\nu')\zeta_\nu(\beta,z,z') - \hat{z}\hat{z}f_\nu f_\nu'\delta(z-z') + (\nabla f_\nu\hat{z}f_\nu' - \hat{z}f_\nu\nabla' f_\nu')\frac{\partial}{\partial z}\zeta_\nu(\beta,z,z')\Big] + W_{\psi\nu}k^2\eta_\nu^{-2}(\nabla\psi_\nu\times\hat{z})(\nabla'\psi_\nu'\times\hat{z})\zeta_\nu(\gamma,z,z')\Big\} \tag{36a}$$

$$\overline{X}_m(\boldsymbol{r},\boldsymbol{r}')=\sum_\nu\left\{W_{f\nu}\left[\hat{z}f_\nu(\nabla' f'_\nu\times\hat{z})\zeta_\nu(\beta,z,z')+\xi_\nu^{-2}\nabla f_\nu(\nabla' f'_\nu\times\hat{z})\frac{\partial}{\partial z}\zeta_\nu(\beta,z,z')\right]+W_{\psi\nu}\left[(\nabla\psi_\nu\times\hat{z})\hat{z}\psi'_\nu\zeta_\nu(\gamma,z,z')-\eta_\nu^{-2}(\nabla\psi_\nu\times\hat{z})\nabla'\psi'_\nu\frac{\partial}{\partial z}\zeta_\nu(\gamma,z,z')\right]\right\} \quad (36b)$$

where the primed functions f'_ν and ψ'_ν are defined with respect to the source point transverse coordinates ρ'. The dyadics $\overline{Y}_e$ and $\overline{X}_m$ can be rewritten in terms of TE and TM modes with the help of (20)–(23)

$$\overline{Y}_e(\boldsymbol{r},\boldsymbol{r}')=\sum_\nu\left\{\frac{k^2}{2j}\left[\beta_\nu^{-1}W_{N\nu}\boldsymbol{N}_\nu^+(\boldsymbol{r})\boldsymbol{N}_\nu^-(\boldsymbol{r}')+\gamma_\nu^{-1}W_{M\nu}\boldsymbol{M}_\nu^+(\boldsymbol{r})\boldsymbol{M}_\nu^-(\boldsymbol{r}')\right]u(z-z')+\frac{k^2}{2j}\left[\beta_\nu^{-1}W_{N\nu}\boldsymbol{N}_\nu^-(\boldsymbol{r})\boldsymbol{N}_\nu^+(\boldsymbol{r}')+\gamma_\nu^{-1}W_{M\nu}\boldsymbol{M}_\nu^-(\boldsymbol{r})\boldsymbol{M}_\nu^+(\boldsymbol{r}')\right]u(z'-z)-W_{f\nu}\hat{z}f_\nu(\rho)\hat{z}f_\nu(\rho')\delta(z-z')\right\} \quad (37a)$$

$$\overline{X}_m(\boldsymbol{r},\boldsymbol{r}')=\sum_\nu\frac{k}{2j}\left\{\left[\beta_\nu^{-1}W_{N\nu}\boldsymbol{N}_\nu^+(\boldsymbol{r})\boldsymbol{F}_\nu^-(\boldsymbol{r}')+\gamma_\nu^{-1}W_{M\nu}\boldsymbol{M}_\nu^+(\boldsymbol{r})\boldsymbol{K}_\nu^-(\boldsymbol{r}')\right]u(z-z')+\left[\beta_\nu^{-1}W_{N\nu}\boldsymbol{N}_\nu^-(\boldsymbol{r})\boldsymbol{F}_\nu^+(\boldsymbol{r}')+\gamma_\nu^{-1}W_{M\nu}\boldsymbol{M}_\nu^-(\boldsymbol{r})\boldsymbol{K}_\nu^+(\boldsymbol{r}')\right]u(z'-z)\right\}. \quad (37b)$$

The dyadics $\overline{Y}_m$ and $\overline{X}_e$ are similar to (37). After the substitution of (37) into (7), we obtain explicit expressions for the electromagnetic field generated in a waveguide by a given distribution of electric and magnetic current densities $\boldsymbol{J}_e$ and $\boldsymbol{J}_m$,

$$\boldsymbol{E}(\boldsymbol{r})=-\frac{1}{2}\sum_\nu\left\{\beta_\nu^{-1}(\omega\mu J_{eN\nu}^- - jkJ_{mF\nu}^-)\boldsymbol{N}_\nu^+(\boldsymbol{r})+\gamma_\nu^{-1}(\omega\mu J_{eM\nu}^- - jkJ_{mK\nu}^-)\boldsymbol{M}_\nu^+(\boldsymbol{r})+\beta_\nu^{-1}(\omega\mu J_{eN\nu}^+ - jkJ_{mF\nu}^+)\boldsymbol{N}_\nu^-(\boldsymbol{r})+\gamma_\nu^{-1}(\omega\mu J_{eM\nu}^+ - jkJ_{mK\nu}^+)\boldsymbol{M}_\nu^-(\boldsymbol{r})\right\}-(j\omega\epsilon)^{-1}\hat{z}J_{ez} \quad (38a)$$

$$\boldsymbol{H}(\boldsymbol{r})=-\frac{1}{2}\sum_\nu\left\{\beta_\nu^{-1}(\omega\epsilon J_{mF\nu}^- + jkJ_{eN\nu}^-)\boldsymbol{F}_\nu^+(\boldsymbol{r})+\gamma_\nu^{-1}(\omega\epsilon J_{mK\nu}^- + jkJ_{eM\nu}^-)\boldsymbol{K}_\nu^+(\boldsymbol{r})+\beta_\nu^{-1}(\omega\epsilon J_{mF\nu}^+ + jkJ_{eN\nu}^+)\boldsymbol{F}_\nu^-(\boldsymbol{r})+\gamma_\nu^{-1}(\omega\epsilon J_{mK\nu}^+ + jkJ_{eM\nu}^+)\boldsymbol{K}_\nu^-(\boldsymbol{r})\right\}-(j\omega\mu)^{-1}\hat{z}J_{mz} \quad (38b)$$

where

$$J_{eN\nu}^-=W_{N\nu}\int_V\boldsymbol{N}_\nu^-(\boldsymbol{r}')\cdot\boldsymbol{J}_e(\boldsymbol{r}')u(z-z')\,dV'$$

$$J_{eN\nu}^+=W_{N\nu}\int_V\boldsymbol{N}_\nu^+(\boldsymbol{r}')\cdot\boldsymbol{J}_e(\boldsymbol{r}')u(z'-z)\,dV',\ \text{etc.}$$

Expressions (38) show that outside the source region the fields are described by expansions of TM (vectors $\boldsymbol{N}_\nu,\boldsymbol{F}_\nu$) and TE ($\boldsymbol{M}_\nu,\boldsymbol{K}_\nu$) modes. The curlless vectors are responsible only for the additional axial field components inside the source region. Expressions (38) are suitable for the separate computation of the amplitude and complex power of each mode, propagating or evanescent. That enables to find the load impedance of the exciting sources, including apertures. The electric and magnetic fields given by expansions (38) satisfy Maxwell's curl (1) and divergence (9) equations. This can be shown by direct substitution.

IV. Green's Functions for Cavities

A cylindrical cavity is a waveguide terminated at $z=0$ and $z=d$ by perfectly conducting walls. The electric and magnetic fields in the cavity can be expanded in terms of the eigenvectors (12) and (15) defined in Section II. The four scalar eigenfunctions of the cavity are related to the waveguide two-dimensional eigenfunctions f_ν and ψ_ν of (19)

$$f_{L\nu}=f_\nu\sin r_s z \qquad f_{M\nu}=\psi_\nu\sin r_s z$$
$$f_{N\nu}=f_\nu\cos r_s z \qquad f_{C\nu}=\psi_\nu\cos r_s z \quad (39)$$

where $r_s=s\pi/d$, s-integer.

From (11) and (16) follows that the eigenvalues of all the cavity eigenfunctions, scalar and vector alike, are

$$k_\nu^2=\xi_\nu^2+r_s^2=\omega_\nu^2\epsilon\mu \qquad (\text{"electric" modes})$$

or

$$k_\nu^2=\eta_\nu^2+r_s^2=\omega_\nu^2\epsilon\eta \qquad (\text{"magnetic" modes}) \quad (40)$$

where ω_ν is the resonant frequency of the respective mode.

These eigenvalues are real numbers because we assumed that the walls are perfectly conducting, thus providing homogeneous boundary conditions. However, real cavities have some internal losses, and are usually loaded. The Q factor of a real cavity accounts for all the losses, which can be ascribed to the medium inside the cavity, supposing that it has some finite conductivity.

The analysis of the free damped oscillations and of the forced oscillations of such a cavity leads to a complex value for k

$$k^2=\omega^2\epsilon\mu(1-j\omega_\nu/\omega Q_\nu) \quad (41)$$

where ω is the frequency of the driving source, and Q_ν is the value of the Q factor for the ν resonant mode.

Now we can write the "electric" and "magnetic" Green's functions, which satisfy (4) and are finite at resonant frequencies

$$\overline{G}_e(\boldsymbol{r},\boldsymbol{r}')=\sum_\nu(k_\nu^2-k^2)^{-1}(W_{L\nu}\boldsymbol{L}_\nu\boldsymbol{L}'_\nu+W_{M\nu}\boldsymbol{M}_\nu\boldsymbol{M}'_\nu+W_{N\nu}\boldsymbol{N}_\nu\boldsymbol{N}'_\nu) \quad (42)$$

where the primed vectors are defined with respect to the source point coordinates r'. The expression for $\bar{G}_m$ is written in a similar way using the eigenvectors (15). Since cavities' eigenvectors are real functions, there is no need for stars to denote conjugate values as in the Green's functions for waveguides (25).

The normalization constants are

$$W_{L\nu}=d^{-1}\epsilon_s W_{f\nu},\ W_{C\nu}=d^{-1}\epsilon_s W_{\psi\nu} \qquad \text{(curlless)}$$

$$W_{N\nu}=W_{F\nu}=\xi_\nu^{-2}k_\nu^2 W_{L\nu} \qquad \text{(TM)}$$

$$W_{M\nu}=W_{K\nu}=\eta_\nu^{-2}k_\nu^2 W_{C\nu} \qquad \text{(TE)} \tag{43}$$

where $W_{f\nu}$ and $W_{\psi\nu}$ are the normalization constants defined by (29), and

$$\epsilon_s=\begin{cases}1, & s=0\\ 2, & \text{otherwise}\end{cases} \tag{44}$$

The dyadics $\bar{Y}$ and $\bar{X}$ which determine the electromagnetic field (7) in the cavity, are found after substituting (42) into (8).

$$\bar{Y}_e=\sum_\nu\Big[-W_{L\nu}\boldsymbol{L}_\nu\boldsymbol{L}'_\nu + k^2(k_\nu^2-k^2)^{-1}\cdot(W_{M\nu}\boldsymbol{M}_\nu\boldsymbol{M}'_\nu+W_{N\nu}\boldsymbol{N}_\nu\boldsymbol{N}'_\nu)\Big].$$

Using the identity

$$k^2(k_\nu^2-k^2)^{-1}=k_\nu^2(k_\nu^2-k^2)^{-1}-1$$

and the "completeness relation" (14) we obtain the final expression for the dyadic $\bar{Y}_e(\boldsymbol{r},\boldsymbol{r}')$

$$\bar{Y}_e=\sum_\nu k_\nu^2(k_\nu^2-k^2)^{-1}(W_{M\nu}M_\nu M'_\nu+W_{N\nu}N_\nu N'_\nu)-\bar{I}\delta(r-r'). \tag{45}$$

Expression (45) shows that in cavities the curlless vectors $\boldsymbol{L}_\nu$ and $\boldsymbol{C}_\nu$ are responsible only for the fields (electric and magnetic) in the source regions. The fields outside the source regions are fully described by the TE and TM modes, i.e., by the eigenvectors $\boldsymbol{M}_\nu$ and $\boldsymbol{N}_\nu$ (electric field) and $\boldsymbol{F}_\nu$ and $\boldsymbol{K}_\nu$ (magnetic field).

The dyadic $\bar{X}_m(\boldsymbol{r},\boldsymbol{r}')$ is

$$\bar{X}_m=\sum_\nu k_\nu(k_\nu^2-k^2)^{-1}(W_{N\nu}\boldsymbol{N}_\nu\boldsymbol{F}'_\nu+W_{M\nu}\boldsymbol{M}_\nu\boldsymbol{K}'_\nu). \tag{46}$$

The dyadics $\bar{Y}_m$ and $\bar{X}_e$ that determine the magnetic field in (7) can be found in a similar way.

After the substitution of (45), (46) into (7), we obtain explicit expressions for the fields generated in a cavity by a given (or assumed) distribution of the electric and magnetic current densities $\boldsymbol{J}_e$ and $\boldsymbol{J}_m$

$$\boldsymbol{E}(\boldsymbol{r})=\sum_\nu k_\nu(k_\nu^2-k^2)^{-1}\{[(j\omega\epsilon)^{-1}k_\nu J_{eM\nu}-J_{mK\nu}]\boldsymbol{M}_\nu(\boldsymbol{r}) + [(j\omega\epsilon)^{-1}k_\nu J_{eN\nu}-J_{mF\nu}]\boldsymbol{N}_\nu(\boldsymbol{r})\}-(j\omega\epsilon)^{-1}\boldsymbol{J}_e(\boldsymbol{r}) \tag{47a}$$

$$\boldsymbol{H}(\boldsymbol{r})=\sum_\nu k_\nu(k_\nu^2-k^2)^{-1}\{[(j\omega\mu)^{-1}k_\nu J_{mK\nu}+J_{eM\nu}]\boldsymbol{K}_\nu(\boldsymbol{r}) + [(j\omega\mu)^{-1}k_\nu J_{mF\nu}+J_{eN\nu}]\boldsymbol{F}_\nu(\boldsymbol{r})\}-(j\omega\mu)^{-1}\boldsymbol{J}_m(\boldsymbol{r}) \tag{47b}$$

where

$$J_{eM\nu}=\int_V \boldsymbol{M}_\nu(\boldsymbol{r}')\cdot\boldsymbol{J}_e(\boldsymbol{r}')\,dV, \text{ etc.} \tag{48}$$

The fields (47) satisfy Maxwell's equation (1) and (9), which is readily shown by direct substitution.

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